IMPLICIT AND EXPLICIT LINEAR SYSTEMS USE IN THE SOLIVNG OF WORD PROBLEMS WITH ALGEBRA BY 14 TO 15 YEAR-OLD STUDENTS

by

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1. Introduction

Solving linear systems without a textual component seemed at first, quite mathematical and logical but devoid of meaning. The difficulties that students face in the mathematisation of word problems using linear systems seemed more important and vital to recognizing how these errors appeared. The mathematisation of a word problem or the translation of a word problem into numerical or algebraic code sometimes leads to the use of algebra in the problem solving process. We therefore were interested in seeing to what extent the mathematisation of word problems is presented in government programmes, specifically in Quebec.

We were motivating in finding out how students solve word problems (sometimes using one or many equations and variables). The central goal of our work was to find out how students solved word problems that seemed to require multiple equations and variables. Take the following problem and its typical solution:

"At a farm, there are only chickens and rabbits. If we count 75 heads and 210 legs, how many chickens and rabbits are there? "
(Translated by the author, Dimensions 2, p.127)

$$\begin{cases} x+y=75 \\ 2x+4y=210 \end{cases} \rightarrow \begin{cases} y=75-x \\ 2x+4y=210 \end{cases}$$

$$\mathbf{2x+4(75-x)=210}$$

$$2x+300-4x=210$$

$$\frac{\text{-}2x}{\text{-}2}=\frac{\text{-}90}{\text{-}2}$$

$$x=45 \text{ poules,}$$

$$y=75-45$$

$$y=30 \text{ lapins}$$

Are we surprised that this word problem appeared in a Sec. II textbook? Obviously, the authors expected a simpler solution, with a single unknown. However, we wondered how Sec. II students could solve such difficult word problems in this fashion.

To address this question, we propose to take a look at certain elements of government programmes which led us to bring up two important questions regarding the mathematisation of word problems at the Sec. II level. We worked with the hypothesis that students used ILS (Implicit Linear Systems) & ELS (Explicit Linear Systems) to mathematise word problems. Others have examined the mathematisation of word problems with single or several unknowns while our proposition may better represent how students mathematise and solve word problems by characterising the type of substitution at play. An experiment has been developed where four carefully selected word problems were given to Sec. II, III & IV students as to see how they are mathematised and solved.

2. Our Questioning Path

At every year of his or her schooling, students develop their ability in solving word problems with algebra. While the mathematisation of word problems occurs in Sec. II, we were surprised to see in the 1994 programme that:

> … students should be given real-life problems whose solution involves generating and manipulating first-degree equations. The situations may be expressed using more than one unknown, but the students must be able to convert them into equations containing one unknown that are of the form ax + b = cx + d.[1]

Not only do they speak of 'more than one unknown', they talk about conversion ability that seems at first, quite a drastic change from the 1981 programme that did not have this additional sentence2. While being an innovative feature, students may now be asked to solve word problems using more than one unknown and be able to convert such situations into this symmetrical equation (possessing an unknown on either side of the equal sign).

Two questions therefore emerged next as to describe the mathematisation of word problems. Is the possibility to use multiple unknowns solicited amongst students? That is, do textbooks present such word problems? Secondly, what tools are offered to the student by textbooks and government programmes regarding the mathematisation of such word problems.

In regards to the first question, we found that textbooks indeed presented word problems that solicited several unknowns. According to Patricia Marchand, Well-Type Comparison word problems[3] (as originally defined by Nadine Bednarz[4]) have two values as originators and students can certainly see two different unknowns. As well, Transformation word problems can also be seen as possessing characteristics that evoke a several unknown mathematisation[5]. Transformation word problems are those where one of the relationships bears a sum or difference as in the following:

> *Well-Type Comparison Word Problem:*
> The badminton team has collected 130$ less than the Handball team and 205$ less than the volleyball team. If the Volleyball team has collected as much as both the Badminton and Handball teams united, how much did each team collect?[6]
>
> *Transformation Word Problem:*
> Yesterday, Christopher had $3.50 less than Marie-Hélène. Today, he has doubled his money while Marie-Hélène's money has increased by $10.40. If both have now the same amount of money, how much did each have yesterday?[7]

While such word problems do appear in Sec. II textbooks, they are plentiful in some cases; other comparison word problems are more abundant. Nevertheless, we need to ask ourselves if textbooks and government programmes provide anything to help students mathematise and solve word problems that seem to require a several unknown solution. In Sec. II textbooks, word problem resolution is done with a single unknown. Often textbooks will indicate that they must choose the unknown to make the mathematisation easier. They might mention assign the letter '*x*' to the first element mentioned, the one that is the least mentioned, or alternately, the one that is present in the question statement. While this advice is often erroneous and misleading, the possibility of several unknowns is evoked. However, textbooks do not present any real tools to convert situations that seem to possess several unknowns to an equation with a single unknown.

We therefore ask ourselves: How do students carry out this task? Must Sec. II students do implicitly what Sec. IV students do explicitly regarding the mathematisation of word problems?

3. Our Working Hypothesis

Confronted by word problems that need to be converted utilising several unknowns, and without any tangible tools that may have been presented by textbooks or government programmes, we present the following hypothesis:

In the mathematisation of such word problems, the student will need to carry out implicitly certain transformations that we define in the following way:

Students use an implicit linear system (ILS) in opposition to an explicit linear system (ELS) where the transformations required in the conversions are carried out implicitly or explicitly.

As to test the pertinence of the introduction of the notions of ILS & ELS, we examined relevant studies.

4. Two Studies

In Radford's 1994 study entitled: *From algebra with a single unknown to algebra with two unknowns*, we see that the "The emergence of algebra of two unknowns seems connected to the necessity of solving problems for which the calculations of a single unknown would prove too difficult."[8] The student needs to transform relations when mathematising and solving word problems that are susceptible to be solved with more than one unknown. This transformation is considered relatively hidden for the student and may not always be expressed symbolically.

In Bednarz', Janvier & Radford's 1995 study entitled: Algebra as a Problem Solving Tool: One Unknown or Several Unknowns ?, we find that they propose that the relative presence of certain word problems (for example, Comparison-Well Word problems) will signal an engagement with one ore several unknowns. The number of unknown may be linked to the structure of the word problem. When they a single unknown to represent the unknown quantities of a situation, their view is that the problem needs to be transformed. However, certain word problems do not require a transformation as we can see in the following:

> *One Unknown Word Problem:*
> Find a number that once doubled and increased by 6, gives the same result that if 13 is removed from its quadruple?[9]

In this word problem, there are no perceived transformations (either implicit or explicit). However, an implicit transformation seems pertinent to characterise student mathematisation while we recognise that this transformation is not apparent when using a single unknown and is hidden and not always expressed symbolically. We therefore propose that the notions of ILS & ELS be used to describe student's mathematisation and solutions with more precision since it takes into account the implicit nature of transformations and permits us to distinguish word problems from those that we name : 'One Unknown Word Problems' (or OUQ).

5. ILS & ELS

We say that one uses an **Implicit Linear System (ILS)** when the system of relationships of a word problem is mathematised into a single equation consisting of a single unknown where one relationship is implicitly substituted into another.

While the ILS cannot describe OUQ word problem that require no transformation, it relates to a situation where there are one or many implicit substitutions. These substitutions are based on the 'canonical system'. Let us look at the following example and its canonical system:

"If Mary and Julie have 12 marbles and
if Julie possesses twice as many marbles
as Mary, how many marble does each have?"

$$\begin{cases} x+y=12 \\ y=2x \end{cases} \rightarrow \; x+\overset{\swarrow 2x}{y} = 12 \;\rightarrow\; x+2x=12$$

An implicit substitution transforms the canonical system (that is, the systemic representation that is the closest to the text of the word problem) by a logical application of a series of relations.

We say that one uses an **Explicit Linear System (ELS)** when a word problem is mathematised explicitly into a system of equations having different symbols (or letters), each representing consistently the same unknown quantity in the word problem, as well, each equation representing the conversion of a relation originating from the word problem. In the following table we may see the difference between stages in the ILS or ELS mathematisation and solution to the previous word problem.

		ILS Stages	ELS Stages
MATHEMATISATION	1. Labelling the unknowns:	# of Julie's marbles: $2x$ # of Marie's marbles: x	# of Julie's marbles: x # of Marie's marbles: y
	2. Transformation:	implicit substitution ↓	None
	3. Equation or System Validation:	Various attempts at creating an equation	Various attempts at creating a system of equations
	4. Equation or System Formulation:	$x+2x=12$ ↓	$\begin{cases} y=2x \\ x+y=12 \end{cases}$ ↓
SOLVING PROCESS:	5. Transformation:	None	explicit substitution ↓ $x+\overset{\swarrow 2x}{y}=12$
	6. Resolution:	$x+2x=12$ $\frac{3x}{3}=\frac{12}{3}$ $x=4$ ↓	$x+2x=12$ $\frac{3x}{3}=\frac{12}{3}$ $x=4$ ↓
	8. Answer:	$x=4$ Marie has 4 marbles and Julie has 8 marbles.	

As we can see, the existence of transformations is prevalent in the ILS mathematisation of a word problem while it is evident in the solving process of an ELS.

Knowing that certain word problems originating from Sec. II textbooks are very similar to those from Sec. IV textbooks and considering that government programmes (and textbooks) indicate that these need to be solved explicitly with a system of equations, how do Sec. II students solve these word problems with a single unknown? How do they realise what we consider to be a hidden substitution within the implicit system? Does the ILS imply a substitution?

6. Methodology and A-Priori Analysis

To answer these questions, we devised an experiment consisting of a questionnaire that is composed of four word problems. Instructions regarding the type of reasoning to be used (arithmetic, algebraic or other) were not provided. Given during class time, students had 30 minutes to answer all questions using their calculator if needed. While the experiment took place in May 2000, interviews with certain students were anticipated but impossible to perform because of school constraints. The test was given to 168 students of l'école secondaire Des Sources (Commission scolaire Marguerite-Bourgeoys). All students came from regular math classes (enriched classes did not take part of this experimentation). While four word problems were given to all students, Sec. III & IV students received only two of the four (in a random fashion).

3 classes of Sec. II	: Math 216	: 60 students	: 240 productions
2 classes of Sec. III	: Math 314	: 64 students	: 128 productions
2 classes of Sec. IV	: Math 416	: 44 students	: 88 productions

These four word problems possessed different structures and we regarded them as susceptible to be mathematised with more than one unknown to be able to answer our research questions. Four elements of criteria were used: I) Connectiveness, II) Quantity of relations and type (additive, multiplicative, …), III) Quantity of knowns, unknowns, totals, constants and non-unary coefficients as well as IV) the Type of quantities (whole, integers, rationals or units). Here are the four word problems:[10]

> <u>‘Exam’ WP</u>
> Jean-François a 2h pour faire les 12 problèmes d’un examen. Il prévoit consacrer 10 min. à chacun d’eux. En réalité, il passe deux fois plus de temps que prévu sur certains problèmes difficiles et deux fois moins de temps sur ceux qu’il trouve faciles. Il termine l’examen un quart d’heure avant l’heure limite. *Combien y avait-il de problèmes de chaque type ?*[11]
>
> <u>‘Candies’ WP</u>
> Marie-Ève et Pat vendent des friandises pour les scouts et les guides. Marie-Ève a vendu 15 boîtes de chocolats et 4 boîtes de bonbons et elle a rapporté 84$. Pat a vendu 24 boîtes de chocolats et 2 boîtes de bonbons pour un total de 108$. *Quel est le prix d’une boîte de chocolats et d’une boîte de bonbons ?*[12]
>
> <u>‘Trees’ WP</u>
> L’érable de 135 cm croît au rythme de 15 cm par an. Le pin de 75 cm croît de 20 cm par an. *Dans combien d’années les deux arbres auront la même taille ?*[13]
>
> <u>‘Students’ WP</u>
> Dans une classe, le nombre de garçons est le double du nombre de filles. La somme des filles et du double du nombre de garçons donne 30 élèves. *Combien y a-t-il de garçons dans cette classe ?*[14]

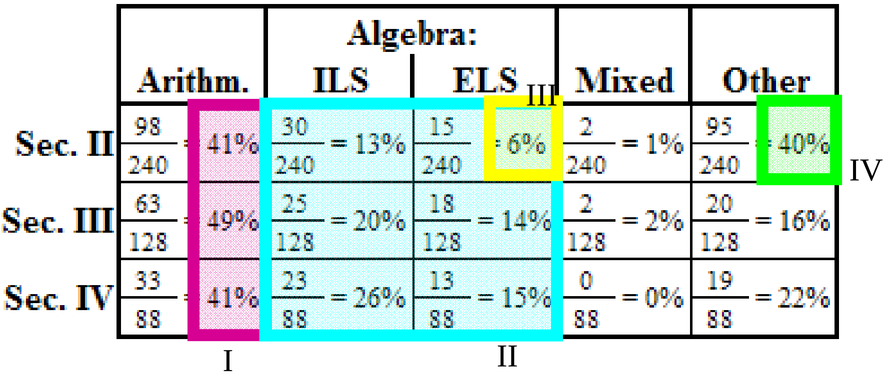
Responses	
Solutions	Other Responses
Arithmetic	Data
Algebraic	Copied
ILS	Extraneous
ELS	Empty
Mixed	Other
Other	

We anticipated different types of mathematisations as indicated in the table at the right. For each of the word problems, a basic system (canonical) is presented in the next table. As well, the number of non-unary coefficients (for each equation), the type of answers, a characterisation of the translations from the original word problem, followed by a description of the type of substitution required in the mathematisation of each of the word problems.

	Exam (a Sec. II Rate WP)	Candies (a Sec. IV Rate WP)	Trees (a Sec. IV Rate WP)	Students (a Sec. II Comparison WP)
Basic System:	$\begin{cases} x+y=12 \\ 20x+5y=105 \end{cases}$	$\begin{cases} 15x+4y=84 \\ 24x+2y=108 \end{cases}$	$\begin{cases} y_1=15x+135 \\ y_2=20x+75 \end{cases}$	$\begin{cases} y=2x \\ x+2y=30 \end{cases}$
Non-Unary Coefficients:	$\begin{cases} 0 \\ 2 \end{cases}$	$\begin{cases} 2 \\ 2 \end{cases}$	$\begin{cases} 1 \\ 1 \end{cases}$	$\begin{cases} 1 \\ 1 \end{cases}$
Integer Answers	Yes	Not Necessarily	Probably	Yes
Direct Translation:	Indirect	of equations	of expressions	of equations
Substitution (substitute → object):	Indirect (SOU): Alg. expression → Unknown of Term	Indirect (SOU) Alg. expression → Unknown of Term	Direct (SOT) Alg. expression → Term	Direct (SOU) Term → Unknown of Term

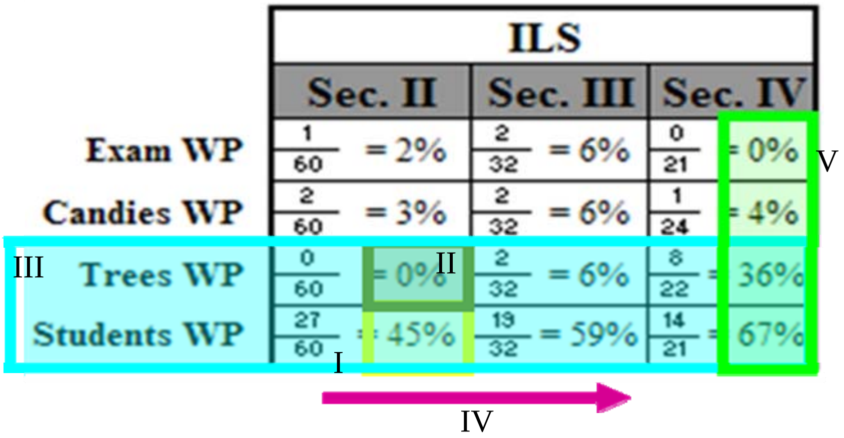
7. Results

When looking at the results of our experimentation, we received a great quantity of arithmetic resolutions (I) but discovered certain progressions in the algebraic resolutions (II). A total of 6% of Sec. II students used an ELS (III) which is quite surprising since they have not previously been exposed to this type of mathematisation. As well, a great deal of Sec. II students used other types of mathematisation (IV).



		Algebra:			
	Arithm.	ILS	ELS	Mixed	Other
Sec. II	$\frac{98}{240} = 41\%$	$\frac{30}{240} = 13\%$	$\frac{15}{240} = 6\%$	$\frac{2}{240} = 1\%$	$\frac{95}{240} = 40\%$
Sec. III	$\frac{63}{128} = 49\%$	$\frac{25}{128} = 20\%$	$\frac{18}{128} = 14\%$	$\frac{2}{128} = 2\%$	$\frac{20}{128} = 16\%$
Sec. IV	$\frac{33}{88} = 41\%$	$\frac{23}{88} = 26\%$	$\frac{13}{88} = 15\%$	$\frac{0}{88} = 0\%$	$\frac{19}{88} = 22\%$

If we look at the word problem breakdown of ILS use, we see that 45% of Sec. II students used an ILS in response to the ‘Students’ WP (I). However, Sec. II student rarely used an ILS with the ‘Trees’ WP (II). If we examine ILS use according to grade level, we see that it increases for the ‘Trees’ and ‘Students’ WP (III). In general terms, we see that Sec. III students use ILS a bit more often than their Sec. II counterparts. (IV) In Sec. IV, ILS are present in substantial numbers, especially in relation to the ‘Trees’ and ‘Students’ WP (V).



	ILS		
	Sec. II	Sec. III	Sec. IV
Exam WP	$\frac{1}{60} = 2\%$	$\frac{2}{32} = 6\%$	$\frac{0}{21} = 0\%$
Candies WP	$\frac{2}{60} = 3\%$	$\frac{2}{32} = 6\%$	$\frac{1}{24} = 4\%$
Trees WP	$\frac{0}{60} = 0\%$	$\frac{2}{32} = 6\%$	$\frac{8}{22} = 36\%$
Students WP	$\frac{27}{60} = 45\%$	$\frac{19}{32} = 59\%$	$\frac{14}{21} = 67\%$

7.1. ILS mathematisations:

There were many notable ILS mathematisations that were quite interesting. Specifically, we found some students able to expose their substitutions. In the first solution, we see that a single equation was used in representing the relationships of the word problem. The quantity ‘number of boys’ is replaced by the term ‘2*x*’ which is then multiplied by 2 (using parenthesis). By adding this to the original ‘*x*’, we get the term ‘5*x*’. In the second solution (215.36), the ‘2*x*’ is

Students : a : 216.20 :
ILS :(T,S,,,EC)

$$x + 2(2x) = 30$$
$$5x = 30$$
$$30 \div 5 = 6$$
$$3x + 2x = 30$$
$$3 \times 6 + 2 \times 6 = 30$$

Students : a : 215.36 :
ILS :(ET,S,,C,)

$$\text{Garçons} = 2x^{\cdot 2}$$
$$\text{Filles} = x$$
$$x + \overbrace{2x^{\cdot 2}}^{4x} = 30$$
$$\frac{5}{5}x = \frac{30}{5}$$

6 filles, 12 garçons

$18 \text{ et } + 12 = 30$ pour le double du nombre de garçons multiplied by two (giving the term ‘4*x*’) and is also combined with the ‘*x*’. While the doubling may not be in the same order for both of these solutions, the substitution is evident. A quantity is substitutde by another without explicit symbolism.

Here a typical solution: In 211.27, we see the equation ‘*x*+2*x* = 30’ is created without regard to the relation which says that the number of boys is the double of the number of girls. Alternately, it is perhaps the other relation that is not fully taken into account. A third possibility emerges when we observe that these two relations have both the same coefficient ‘2’ and that confusion between the both ‘twos’ (or recognition that they are the same) could have occurred.

Students : a : 211.27 : ILS :(ECT,S,,,)

filles : x	$x+2x=30$	Rep :
gars : $2x$	$\frac{3}{3}x=30$	Filles : 10
	$x=10$	$30-10=20$ gars

Student #211.07 solved the ‘Students’ WP by defining the total number (of boys) as being ‘*x*’ and by defining the number of girls as being ‘ $x \div 2$ ’. We can see that the selection of a generating quantity is an essential step in being able to represent successfully the word problem algebraically. His solution is quite unusual, as we have found that most students that have solved this word problem have used the number of girls to be equal to ‘*x*’. The student was able to compose a correct equation. The simplification of this equation is flawed. He firstly commuted the ‘ $(x \div 2)+(x \times 2)$ ’ into ‘ $2x + x \div 2$ ’. He then added ‘2*x*’ with ‘*x*’ but stopped there. The last expression was mistakenly taken as being ‘ $(2x+x)\div 2$ ’. He would have perhaps arrived at the correct answer if he had instead simplified ‘ $2x+\frac{x}{2}$ ’. His algebraic solution is perhaps erroneous yet, he finds an answer to the equation: ‘ $\frac{3x}{2}=30$ ’ by executing opposite operations.

Students : d : 211.07 :
ILS :(ECT,S,,,)

$x =$ de garçons

$x \times 2 =$ n total

$x \div 2 =$ n de fille

$(x \div 2)+(x \times 2)=30$

$2x + x \div 2 = 30$

$3x \div 2 = 30$

$30 \times 2 = 60 \div 3 = 20$

$x = 20$ garçons

Student #211.30 wrote two equations each using the letters ‘*f*’ and ‘*G*’. Both equations seem to be related to the value ‘30’ yet they also seem to contradict themselves. If ‘ $f \times 2 = G$ ’, ‘ $f + f$ ’ (which is equal to $f \times 2$) can not be equal to $G \times 2$. It is quite important and interesting contradiction that many adults we tested fell into: the confusion between ‘ $G = 2 \times f$ ’ and the double of the boys. Clearly, for this student the equations ‘ $f \times 2 = G$ ’ and ‘ $f + f = G \times 2$ ’ are the same. The letters do not seem to be variables but value holders, as they are evidently not manipulated in order to solve the word problem. The ‘30’ is calculated by summing the values of the girls and boys but it is not clear how these two equations are related. What is interesting is that we see a Sec. II student attempt to mathematise the relationships of the word problem utilising two different letters sadly, without being able to combine them into an equation that is solvable. Examples such as this one is evidence that at least as a representational tool, the use of many variables and many equations naturally occurs at this grade level. The letters they use may represent each quantity yet, these students have not been taught how to combine them together.

Students : b : 211.30
ELS :(E,,S,,CT)

$$\left.\begin{array}{r} f \times 2 = G \\ f + f = G \times 2 \end{array}\right\} 30$$

$$\left.\begin{array}{r} \text{Fille} = 10 \\ \text{Garçons} = 20 \end{array}\right) 30 \text{ élèves}$$

Students : b : 311.33 : ELS :(,,S,T,)

$F =$ nb de filles

$G =$ nb de Garçons $= 2 \cdot F$

$F + 2\cancel{G}^{F} = 30$

$6 + 2 \cdot \cancel{6}^{12} = 30$

Réponse : 12 garçons

Here, both letters are defined and an equation is constructed: ‘ $G =$ nb de Garçons $= 2 \cdot F$ ’. We see in the third line that the letter ‘*G*’ is cancelled and replaced by ‘*F*’. It seems that the letter *G* is just cancelled and replaced by the superimposed symbol ‘*F*’ representing all the girls. We may believe that he did not want to write down another ‘2’ coefficient since it already had one. Both sixes may represent the value of the unknown. He then multiplies it by two to get twelve. This student’s solution is a bit more advanced than the preceding one since not only are there two different letters present, his two-lettered equation is altered to incorporate the first relationship (namely that there are twice as many boys as girls).

In the next student solution, we see that the correct solution is also found. The student writes the equation ‘ $2g+f=30$ élèves ’ which is a bit odd since we expected the equation: ‘ $f+2g=30$ ’ (a literal transcription of the second relationship of the word problem). The ‘substitution’ step is missing, since he goes directly from this equation to the division of thirty by five that leads him to find the correct answer.

Students : b : 215.06 : ELS :(ET,,S,,C)

Dévellopement (g) gars : $2f=g=12$ gars

(f) fille : $f=6$ fille

total d'élève : $2g+f=30$ élèves

réponse: garçons : 12

filles : 6

$$\begin{array}{r|l} 30 & 5 \\ \cline{2-2} -30 & 6 \\ \hline 0 & \end{array}$$

We believe that the ‘=12 gars’ & the ‘=6 fille’ were added once the solution to the problem was found. We can not help but wonder why he does not write down the critical element, that ‘2*g*’ transforms into ‘4*f*’ and that ‘ $4f+f$ ’ becomes ‘ $5f$ ’. These details may be regarded as too elemental and have been performed mentally and perhaps not made explicit.

Here is another student who seems to have decided that the ‘30’ must be partitioned into 10 & 20. He firstly writes two equations that literally represent the word problem’s two relationships and then writes the equality: ‘ $30-20=10$ ’ but affixes the words ‘gars’ & ‘fille’ atop the ‘20’ & ‘10’. This may indicate that the twenty is considered the total number of boys and that the thirty is simply the total number of students. This student uses at first ELS symbolism but reverts to arithmetic when actually solving. Without an interview, we can only assert that his arithmetic verification has an algebraic feel.

Students : a : 413.22 : ELS :(E,,S,,)

$1g=2f$

$1F+2g=30$

$30-20^{gars}=10^{fille}$

$10^{fille}\times 2=20^{garc}$

$10^{fille}+20^{garc}=30$

$\left\langle\begin{matrix}10\text{ filles}\\ 20\text{ gars}\end{matrix}\right\rangle$ réponse

Why perform so many checks? They may signal certain apprehensions in the mathematisation process, that is, difficulty in justifying his answers with the quantities of the word problem. Still, he may also have certain confusion regarding the association of his answers with the two equations he writes originally. Amazingly, he insists on maintaining some links with the context of the word problem by affixing these labels to numbers. These actions may be also considered pre-cursors to algebraic symbolism. It is surprising to see that this Sec IV student does not master algebraic problem solving considering that this is a classical Sec. II word problem. Additionally, this Sec. IV student, while having received formal training in solving linear systems, prefers an arithmetic solution. It may be that he never needed to write any more algebra since this wrong answer might have been so evident for him.

Students : a : 312.03 : ELS :(,,S,,E)

nombre de gars = 2 fois nombre de filles

$G=2F$

$f+2G=30$

$\xcancel{30=}\text{nombre d'élèves}=30-\frac{\text{ng}}{2}$

$2f+f=\text{nombre d'élèves}$

$20+10=30$

$\overset{\overset{2g}{\uparrow}}{20}+\overset{\text{filles}}{10}=30$

$\overset{\text{nf}\times 2}{\overset{\frown}{10\times 2}}\overset{=\ \text{garçons}}{=\quad 20}\qquad\boxed{\text{rep}:20\ \text{garçons}}$

Similarly to the last solutions, student #312.03 also writes an ELS at the very start of his work. The student crosses-out the thirty since he sees that he must subtract the term ‘ $\frac{\text{ng}}{2}$ ’ (equal to the number of girls – number of boys divided by two) from thirty to obtain the number of students (‘nombre d’élèves). Sadly, the student confuses the concepts: ‘number of boys in the class’ and ‘quantity of boys implicated in this study’. We never mention in the problem that there are thirty boys and girls in the class. As we have mentioned earlier, this student may have seen that since we have the expression ‘ $2f + f$ ’, the numerical trial ‘20+10’ seems to fit. The next few lines come to re-assure him that the previous line is correct.

The following solution is quite hard to follow, let us examine it closely. A first equation is written which states that ‘ $F + 2G = 30$ ’. He then writes the equation ‘ $\frac{F}{2} = G$ ’ but he is unsure if the number of boys is equal to the double or the half of the number of girls. He then changes this to ‘ $2F = G$ ’. Attempting to create an expression that will become the right side of the equation (both start similarly with the term: ‘ $F +$ ’), he crosses-out the ‘ $\frac{F}{2}$ ’ and does not continue. The student assigns a meaning to the letters he uses. He writes the answer 10 girls and 20 boys that must have been calculated either mentally or may have originated from somewhere else. He may have perceived that finishing the algebraic reasoning would be much harder than finding the answer in other ways. He did not see why he should finish his algebraic manipulations if he already knew the answer. It is interesting to see that this student takes the time to cancels-out the wrongful elements. It is difficult to perceive the substitution of ‘ $2F = G$ ’ into the equation ‘ $F + 2G = 30$ ’ (if this was indeed what was done). Ironically, his algebraic reasoning may have contributed to help him find the erroneous 10 / 20 answer. The equality ‘10+20=30’ is very close to the equation ‘F+2G=30’ especially if you believe that F=10 and 2G=the number of boys that is equal to 20. A selection of different numbers in the wording of the problem may have helped us in this regard.

Students : b : 311.06 :
ELS : (, , S , , C)

$F + 2G = 30$ $\quad 2\frac{F}{\cancel{2}} = G$

$F + \cancel{\frac{F}{2}}$ $\quad$ 10 Filles
20 Gars
F

F : Filles
G : Gars

In the following solution, both the ‘*x*’ and ‘*y*’ are defined and help to explain the equations in the third and fourth line. A first equation is written and relates to the first relationship of the word problem. The second equation seems to be constructed through substitution of the first. We see that some students might be confused and consider the ‘doubling’ aspect present in both relationships to be one and the same. The solution seems to support our contention that some students see the phrase ‘double du nombre de garçons’ as a re-iteration of the first statement: ‘le nombre de garçons est le double du nombre de filles. We see algebraic evidence of this since the expression ‘2*y*’ is repeated (and therefore does not need to be substituted). Surprisingly, this Sec. III student does not solve the equation even if he would have solved such equations in Sec. II. He is unable to combine ‘ $2y + y$ ’ together, perhaps because it would lose its meaning relative to the word problem.

Students : b : 311.09 :
ELS : (C , , S , ,)

nombre de garcon = x
nombre de fille = y
$x = 2y$
$30 = 2y + y$
$2y + y = 30$

Here we see that for student #312.02, he mathematises the first relationship into the equation: ‘ $2g = f$ ’ which is the classical representation of the static composition principle defined by Clement (1982). He then states that the total is thirty. This student might have wanted to perform ‘ $2 \times (2g) = 4g$ ’ instead of ‘ $2 \times 2 = 4$ ’. If it follows from ‘ $f + 2g = 30$ ’ that ‘ $30 - 4 = 26$ ’, there should be four girls and the 26 should represent the double of the number of boys. Since he gives us

Students : b : 312.02 :
ELS : (, , S , T ,)

$2g = f$
Total $= 30$
$2 \times 2 = 4$
$f + 2g = 30$
$30 - 4 = 26$
Rép : 26 gars

the answer of 26 boys, this would mean that for him ‘2g’ would correspond to the number of boys and not the double of the number of boys. However, the word problem asks to double the number of boys, this student may have given us the value of ‘2g’ as his final answer.

This solution is quite interesting on many accounts. Not only do we see clear evidence that the Student-Professor problem is evident, we see that for some students the term ‘2*x*’ does not necessarily mean the double of the value of ‘*x*’. The double of a quantity could be an entity itself. Still, this Sec. III student attempted to solve this word problem using an ELS. At this grade level, he has never been taught how to substitute explicitly expressions into others.

We consider the 413.02 solution to be an ILS because of the absence of two different variables and two equations. It could have been regarded as a comparison (the two sizes of trees must be equal). One could consider that the pupil substituted the expressions ‘135+15*x*’ and ‘75+20*x*’ in the relation “Height of maple” is equal to the “Height of the pine”. One anticipated much more Sec. IV ELS mathematisations relatively with the problem of the trees. We have found that Sec. IV students do not write two different letters however, we find that the type of substitution is rather explicit.

Trees : b : 413.02 : ILS :(, ET,,,)

Opé: $135+15x$

$75+20x$

$135+15x^{-15x} = 75+20x^{-15x}$

$135^{-75} = 75^{-75} + 5x$

$60 = 5x$

12 ans

One Letter Representing all of the Word Problem’s Unknown Quantities

In our examination of the ILS mathematisations, we came across many strange equations that we’re meant to express the relationships of the word problem.

In many cases, one letter (the ‘*x*’ in most cases) is used to represent all of the word problem’s unknown quantities. In both 216.31 and 312.22, we see that the letter ‘*x*’ is used to represent the easy and difficult questions (or the number of boxes of chocolates or candies) without differentiation. The pupil simplifies and obtains ‘25*x*’ but does not continue because he must certainly realize that this partition does not work to identify the number of easy and difficult questions. A similar situation was also obtained in the second study[15].

Exam : b : 216.31 : ILS :(, ES,,, CT)

1h30

$20+5$

$\mathbf{20x + 5x = 1h30}$ (90)

$^{40}/_{50}$

$25x = 90$

2 * types de difficile

10 types de facile

Candies : a : 312.22 : ILS :(T, C,,,)

24 chocolats =108$

2 bonbons

$\mathbf{24x + 2x = 108}$

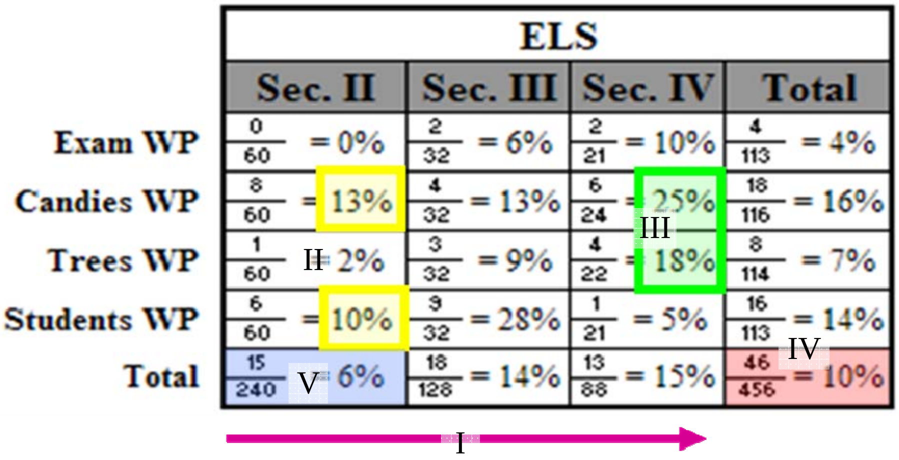
$\cancel{3\times 24 = 72\$}$ $4\times 24 = 96\$$

$\cancel{18\times 2 = \cancel{18}\,36}$ $6\times 2 = 12$

chocolat = $\overset{4}{\cancel{3}}$ $ chacun bonbons = $\overset{6}{\cancel{18}}$ $ chacun

ELS

As expected many students used other algebraic mathematisations, such as the ELS. We found that the ELS use increases at each school year, as indicated in the following diagram (I). 13% and 10% of Sec. II students use ELS for the Candies and Students word problems respectively (II). Sec. IV students used an ELS for Sec. IV word problems (the ‘Candies’ and ‘Trees’ WP) rather than with Sec. II word problems (the other two) (III). While this diagram speaks of total productions, only 39 students out of 168 used an ELS (IV). However, 14 Sec. II students (providing 15 solutions) used an ELS without any instruction in this regard (V).



	ELS			
	Sec. II	**Sec. III**	**Sec. IV**	**Total**
Exam WP	$\frac{0}{60} = 0\%$	$\frac{2}{32} = 6\%$	$\frac{2}{21} = 10\%$	$\frac{4}{113} = 4\%$
Candies WP	$\frac{8}{60} = 13\%$	$\frac{4}{32} = 13\%$	$\frac{6}{24} = 25\%$	$\frac{18}{116} = 16\%$
Trees WP	$\frac{1}{60} = 2\%$	$\frac{3}{32} = 9\%$	$\frac{4}{22} = 18\%$	$\frac{8}{114} = 7\%$
Students WP	$\frac{6}{60} = 10\%$	$\frac{9}{32} = 28\%$	$\frac{1}{21} = 5\%$	$\frac{16}{113} = 14\%$
Total	$\frac{15}{240} = 6\%$	$\frac{18}{128} = 14\%$	$\frac{13}{88} = 15\%$	$\frac{46}{456} = 10\%$

Student #216.27 wrote down two equations without providing a solution. Correct answers are supplied but we noticed that 'garçon = 12' was written before 'fille = 6' which may mean that this student did not substitute as we would have thought. If he had substituted the ' $g = 2f$ ' into the ' $f + 2g = 30$ ', they would have obtained ' $f + 4f = 30$ ' or ' $5f = 30$ ' that can be reduced to ' $f = 6$ '. We would expect them to find the number of girls '*f*' first and then calculated the number of boys. We do not know if the letters present in both equations (each representing a distinct relationship of the WP) are indeed variables and not something else since they are not used in solving the equations. In any case, they did find the right answer but did not give us all the steps in his solution (the crucial 'substitution' step is missing). The use of two equations and two variables to represent the WP is quite impressive considering that it was done by Sec. II student who are quite unaware of ELS solving methods. We have found that once an equation or several equations are laid out, some students will use their calculators in attempting in finding values that satisfy both equations. This may have been the case here.

Students : a : 216.27 :
ELS :(T,,,CS,E)

$g = 2f$
$f + 2g = 30$
garçon = 12
fille = 6

Student 312.27 (a Sec. III student) substitutes repeatedly the 'g = 2f' equation into the '*f*+2*g* = 30' equation. He even goes further by dissociating the '2*f*' into '*f*+*f*' and the '2*g*' into '(*f*+*f*)+(*f*+*f*)'. Curiously, this student could have continued by recombining the 5 '*f*' to solve the problem. Even if he did not continue, his solution gives us insight into how students substitute.

Students : a : 312.27 :
ELS :(E,,S,,)

$2f = g$
$f + 2g = 30$
$f + (f + f) + (f + f) = 30$

In this Sec. II solution, we see that the student has started to translate the problem in a literal way. One can see the repetition of the variable '*x*'. This is perhaps due to the fact that the two unknowns are boxes of 'something'. We may then see that the '*x*' is cancelled and a '*y*' is apposed. This student was then incapable to continue his solution for possibly two reasons: i) he does not know how to handle multiple equations (and multiple letters) or ii) this system is much too complex to solve (a substitution or comparison solution could be out of his reach).

Candies : c : 216.31 :
ELS :(,ES,C,,T)

M – É : $15x + 4\overset{y}{\cancel{x}} = 84$
Pat : $24x + 2\overset{y}{\cancel{x}} = 108$

This student has mathematised the word problem into a system quite well but went one step further, he added the coefficients.
Not knowing really what to make of the resulting equation, he could not continue

Candies : b : 216.27 : ELS :(T,,SC,,E)

$$15a + 4b = 84$$
$$24a + 2b = 108$$
$$39a + 6b = 192$$

Here we see the only correct ELS solution to the ‘Candies’ WP. Surprisingly, this exemplary solution was done by a Sec. II student who has not received any instruction in this regard.

Candies : c - 215.27 : ELS :(ETS,,,C,)

$$\text{M.E} : 15a + 4b = 84$$
$$\text{Pat} : 24a + 2b = 108$$

$$2b = 108 - 24a$$
$$b = \frac{\overset{54}{\cancel{108}} - \overset{12}{\cancel{24}}a}{\overset{1}{\cancel{2}}}$$
$$15a + 4(54 - 12a) = 84$$
$$15a + 216 - 48a = 84$$
$$15a - 48a = 84 - 216 = -132$$
$$-33a = -132$$
$$a = \frac{132}{33} = 4\$$$

$$15 \times 4 + 4b = 84$$
$$60 + 4b = 84$$
$$4b = 84 - 60 = 24$$
$$b = \frac{24}{4} = 6$$

rep: boîte de chocolat : 4\$
boîte de bonbons : 6\$

8. Conclusion

How do Sec. II students solve word problems that are prone to a several unknown solution, we found that surprisingly, many used arithmetic means (41% of Sec. II responses)[16]. They also use an ILS (13% of responses) but surprisingly, they also used an ELS (6% of responses). The ILS responses were often adequate (in form and also as a means to find a correct solution), they sometimes signalled the repeated use of the same unknown for two unknown quantities. Some ILS solutions were incorrect whereas the substitution did not take into account certain coefficients (we can think of the 'double' found in the 'Students' WP).

While there were many ELS Sec. II solutions, many were simply mathematisations into systems and did not provide ELS solutions. We found that 14 of the 60 Sec. II students that were part of the study provided at least one ELS mathematisation but since they did not possess any formal training in solving systems of equations, they for the most part, were not able to provide a complete solution.

In response to the other part of our question, "How do students substitute in an ILS mathematisation?", we found that substitutions are evident when parenthesis or intermediate manipulations (algebraic or arithmetic) are present as we can see in these two examples. These substitutions are difficult to identify and are implicit. They made us examine if we could caractarise further what is being substituted, a term or an unknown (see René de Cotret & Coulange 20xx). As we can see following diagram (relative to the 'Students' WP), we found that for many, the substition of a term is confused with the substitution of an unknown. We can see that the '2y' is substituted by '2x'.

$$\boxed{\begin{array}{r} x+2(2x)=30 \\ 5x=30 \end{array}} \boxed{\begin{array}{c} \text{Garçons} = 2x^{\bullet 2} \\ \text{Filles} = x \\ x+\overbrace{2x^{\bullet 2}}^{4x}=30 \end{array}}$$

$$\begin{cases} y=2x \\ x+2y=30 \end{cases} \rightarrow \quad x+\overbrace{2y}^{\overset{y=2x}{\swarrow}}=30 \quad \rightarrow \quad x+2x=12$$

While the ILS & ELS can be seen as useful to examine the mathematisation and solution to word problems, we have found that the substitution itself could be even further refined therefore increasing its effectiveness in relating to us the student's work.

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[1] Boisvert, L.; Djiknavorian, M.; Boulanger, F. and Lagacé, J. (1994). *Programmes d'études - Mathematics 216, Secondary School (Document #16-3301-07).* Québec: Ministère de l'éducation, Gouvernement du Québec (p. 26).

[2] Québec, Ministère de l'Éducation. (1981). *Programmes d'études - Mathématique, enseignement secondaire premier cycle (Document #16-3301-00).* Québec: Ministère de l'éducation, Gouvernement du Québec (p. 22).

[3] Marchand, P. (1997). Résolution de problèmes en algèbre au secondaire: Analyse de deux approches et des raisonnements des élèves. Unpublished Master's Thesis, Université du Québec à Montréal, Montréal.

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[5] Bednarz, N., Radford, L. and Janvier, B. (1995). Algebra as a problem solving tool: one unknown or several unknowns? Paper presented at the 19th Annual conference of the International group for the psychology of mathematics education, Recife, Brazil.

[6] Guay, S. and Lemay, S. (1994). *Scénarios 2.* Laval: HRW (Translated by the author, p. 55).

[7] Ibid (Translated by the author, p. 183).

[8] Radford, L. (1994). Moving through system of mathematical knowledge: from algebra with a single unknown to algebra with two unknowns. Paper presented at the 18th Annual conference of the International group for the psychology of mathematics education, Lisbon, Portugal (p. 74).

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[9] Guay, S. and Lemay, S. (1994). *Scénarios 2.* Laval: HRW (Translated by the author, Scénarios 2, p. 189)

[10] The for your convenience, here are translations of the four word problems used in the study:

The ‘Exam’ WP: Jean-François has 2 hours to solve the 12 problems of the exam. He forces taking 10 minutes for each of them. In reality, he takes twice as much for certain hard problems and twice as little for the easy ones. He finishes the exam a quarter of hour before the time limit.
How many problems of each type are there?

The ‘Candies’ WP: Marie-Ève and Pat sell candies for the scouts and Girl Guides.
Marie-Ève has sold 15 boxes of chocolates and 4 boxes of candies and has amassed $84.00.
Pat sold 24 boxes of chocolates and 2 boxes of candies for a total of $108.00.
What is the price of a box of chocolate and a box of candies?

The ‘Trees’ WP: The Maple has a height of 135 cm and grows at a speed of 15 cm per year.
The Pine measures 75 cm and grows 20 cm per year. *In how many years will both trees have the same height?*

The ‘Students’ WP: In a classroom, the number of boys is twice the number of girls.
The sum of the number of girls and the double of the number of boys is 30 students.
How many boys are there in this class?

[11] Jordi, I.; Patenaude, P. & Warisse, C. (1994) Dimensions mathématique 216 (Tome 1). Saint-Laurent: ERPI, p. 137 (Problem #19).

[12] Breton, G.; Deschênes, A. & Ledoux, A. (1996) Regards mathématiques 416 (Tome 1). Anjou: CEC, p. 229 (Problem #4). This problem was slightly altered as to explicit the names of the known quantities. In the original problem, there were chocolate bars and rabbits. Some students would not understand that both were chocolate.

[13] Breton, G.; Deschênes, A. & Ledoux, A. (1996) Regards mathématiques 416 (Tome 1). Anjou: CEC, p. 205 (Problem #7).

[14] Breton, G. (1994) Carrousel mathématique 2 (Tome 1). Anjou: CEC, p. 261 (Problem #15). This problem was slightly altered to increase its difficulty. Originally, the problem stated that the sum of boys and girls was thirty.

[15] The problem of false one unknown problems is presented in Bednarz, Radford and Janvier 1995.

[16] While the arithmetic solutions were quite interesting and thought provoking, they could not be presented in this article.